more flexibility in the selection of the approach path between the holding pattern and the final descent on the glidepath.

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Error Analysis Strapdown Inertial Navigation Using Quaternions

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I. Introduction

IN recent years, airborne gimbaled inertial navigation systems (INS) have been replaced with strapdown systems by virtue of the utilization of the ring laser gyro and high-performance microcomputers. Additionally, the conventional hybrid navigation systems, which uses a gimbaled INS, and other navigation systems have also been modified to use a strapdown INS and other navigational aids (for example, DOPPLER, LORAN-C, OMEGA, GPS.

In a strapdown INS the sensors (gyros and accelerometers) are rigidly attached to the body of the vehicle; hence, the quantities they measure (angular velocity, specific force) are in body-fixed coordinates. Determination of the position of the body, however, requires knowledge of the specific force components in the navigation (local vertical) coordinates.

Therefore, the transformation from body-fixed frame to navigation frame requires direction cosine matrix computation based on accurate knowledge of vehicle attitude. For the purpose of this computation, it is desirable to use a quaternion method, which is more advantageous than the Euler angle method or the direction cosine method from the aspect of computational speed and computational accuracy.[1,2]

In Ref. 1, a strapdown inertial navigation system, based on a quaternion including dynamics and error propagation, is mechanized in an inertial reference frame. Inertially referenced, or space-stable, navigation systems are widely used for spacecraft applications where geographic navigation information is not required. But, for terrestrial navigation, the inherent time-varying relationship between the inertial and geographic (local vertical) frames complicates the space-stable system design. Then the local-vertical mechanization is a better choice for terrestrial applications of strapdown navigation systems.

The purpose of this paper is to introduce strapdown inertial navigation error equations based on a quaternion relation between body-fixed frame and navigation (local vertical) frame for terrestrial hybrid navigation systems.

II. Dynamic Equations

The differential equation of the relative quaternion between body-fixed coordinates and local vertical coordinates is given by

$$\dot{q} = ½[\Omega_b]q - ½[\Omega_n]q \tag{1}$$

where q is a four-component vector of "quaternion parameters"

$$q^T \triangleq [q_0, q_1, q_2, q_3]$$

The skew-symmetric matrices $[\Omega_b]$, $[\Omega_n]$ are given by

$$[\Omega_b] = \left[\begin{array}{c|c} 0 & -\omega_b^T \\ \hline \omega_b & [\omega_b] \end{array}\right] = \left[\begin{array}{c|ccc} 0 & -\omega_{bx} & -\omega_{by} & -\omega_{bz} \\ \hline \omega_{bx} & 0 & \omega_{bz} & -\omega_{by} \\ \omega_{by} & -\omega_{bz} & 0 & \omega_{bx} \\ \omega_{bz} & \omega_{by} & -\omega_{bx} & 0 \end{array}\right] \tag{2}$$

$$[\Omega_n] = \left[\begin{array}{c|c} 0 & -\omega_n^T \\ \hline \omega_n & [\omega_n]^T \end{array}\right] = \left[\begin{array}{c|ccc} 0 & -\omega_{nx} & -\omega_{ny} & -\omega_{nz} \\ \hline \omega_{nx} & 0 & -\omega_{nz} & \omega_{ny} \\ \omega_{ny} & \omega_{nz} & 0 & -\omega_{nx} \\ \omega_{nz} & -\omega_{ny} & \omega_{nx} & 0 \end{array}\right] \tag{3}$$

where

$$\omega_b = \begin{bmatrix} \omega_{bx} \\ \omega_{by} \\ \omega_{bz} \end{bmatrix}, \quad \omega_n = \begin{bmatrix} \omega_{nx} \\ \omega_{ny} \\ \omega_{nz} \end{bmatrix} = \begin{bmatrix} \Omega_x + \rho_x \\ \Omega_y + \rho_y \\ \Omega_z + \rho_z \end{bmatrix}$$

$$= \begin{bmatrix} (|\omega_e| + \dot{\Lambda})\cos\lambda \\ -\dot{\lambda} \\ -(|\omega_e| + \dot{\Lambda})\sin\lambda \end{bmatrix} \tag{4}$$

ω_{bx}, ω_{by}, and ω_{bz} are the inertial rate of the body-fixed frame in the body-fixed coordinates, and ω_{nx}, ω_{ny}, and ω_{nz} are the inertial rate of the local vertical frame in the local vertical coordinates. Ω_x, Ω_y, and Ω_z are the Earth rate in the local vertical coordinates; ρ_x, ρ_y, and ρ_z are the rotation rate of local vertical coordinates; $|\omega_e|$ is the Earth rate; Λ is the vehicle longitude, and λ is the verticle latitude.

In terms of the quaternion, the transformation (direction cosine) matrix C_B^N from body-fixed coordinates to navigation coordinates is given by

$$C_B^N = \begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 2(q_0q_3+q_1q_2) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1) \\ 2(q_1q_3-q_0q_2) & 2(q_0q_1+q_2q_3) & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix} \tag{5}$$

It is observed that Eq. (1) is bilinear in ω_b, ω_n, and q, i.e., we can also write Eq. (1) as

$$\dot{q} = ½Q(q)\omega_b - ½R(q)\omega_n \tag{6}$$

where

$$Q(q) = \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix}, \quad R(q) = \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & q_3 & -q_2 \\ -q_3 & q_0 & q_1 \\ q_2 & -q_1 & q_0 \end{bmatrix} \tag{7}$$

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The differential equation of the vehicle velocity vector in the local vertical coordinates is given by

$$\dot{V}_N = C_B^N a_b - [(2\Omega_e+\rho)_n]V_N - [(\Omega_e)_n]^2 \cdot R_n + G$$
$$= C_B^N a_b - [(2\Omega_e+\rho)_n]V_N + g(R)_n \tag{8}$$

$$[(2\Omega_e+\rho)_n] = \begin{bmatrix} 0 & -(2\Omega_z+\rho_z) & (2\Omega_y+\rho_y) \\ (2\Omega_z+\rho_z) & 0 & -(2\Omega_x+\rho_x) \\ -(2\Omega_y+\rho_y) & (2\Omega_x+\rho_x) & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & (2|\omega_e|+\dot{\Lambda})\sin\lambda & -\dot{\lambda} \\ -(2|\omega_e|+\dot{\Lambda})\sin\lambda & 0 & -(2|\omega_e|+\dot{\Lambda})\cos\lambda \\ \dot{\lambda} & (2|\omega_e|+\dot{\Lambda})\cos\lambda & 0 \end{bmatrix} \tag{9}$$

$$R_n = \begin{bmatrix} 0 \\ 0 \\ -(R_E+h) \end{bmatrix}, \quad a_b = \begin{bmatrix} a_{bx} \\ a_{by} \\ a_{bz} \end{bmatrix}, \quad V_N = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$$

$$g(R)_n = \begin{bmatrix} 0 \\ 0 \\ g(1-2h/R_0)\cdot \\ (1+5.2884\times 10^{-3}\sin^2\lambda) \end{bmatrix} \tag{10}$$

$$R_E = R_0(1+E\sin^2\lambda), \qquad R_0 = 2.0925860\times 10^7 \text{ ft}$$
$$E = 1/294.978613, \qquad g = 32.08822 \text{ ft/sec}^2 \tag{11}$$

where a_b is the specific force vector along the body axes, V_N the velocity vector in the local vertical coordinates, and h the vehicle altitude.

Similarly, the differential equations of the vehicle position in the local vertical coordinates are given by

$$\dot{\lambda} = V_x/(R_N+h)$$
$$\dot{\Lambda} = V_y/[(R_E+h)\cos\lambda]$$
$$\dot{h} = V_z \tag{12}$$

where $R_N = R_0(1-2E+3E\sin^2\lambda)$.

III. Error Equations

Equations (1), (8), and (12) give the true attitude and motion based on true specific force and angular velocity.

In an actual navigation system, only indicated quantities are available, namely, q^* = indicated quaternion parameters, ω_b^* = indicated body angular velocity vector, ω_n^* = indicated angular velocity vector of the local vertical coordinates, V_N^* = indicated velocity vector in the local vertical coordinates, a_b^* = indicated specific force vector, and so forth.

The navigation errors may be defined as the differences between the true and indicated quantities.

$$\delta q = q - q^*, \qquad \delta\omega_b = \omega_b - \omega_b^*, \qquad \delta\omega_n = \omega_n - \omega_n^*$$
$$\delta V_N = V_N - V_N^*, \qquad \delta a_b = a_b - a_b^*, \qquad \delta g = g(R)_n - g(R^*)_n$$
$$\delta\Omega = (\Omega_e)_n - (\Omega_e)_n^*, \qquad \delta\lambda = \lambda - \lambda^*, \qquad \delta\Lambda = \Lambda - \Lambda^*, \delta\rho = \rho - \rho^* \tag{13}$$

For sufficiently small errors, products of errors are negligible and, to first order in the error, quaternion errors satisfy the following differential equations.

$$\delta\dot{q} = \tfrac{1}{2}[\Omega_b^*]\delta q - \tfrac{1}{2}[\Omega_n^*]\delta q$$
$$+ \tfrac{1}{2}Q(q^*)\delta\omega_b - \tfrac{1}{2}R(q^*)\delta\omega_n \tag{14}$$

where

$$\delta q = \begin{bmatrix} \delta q_0 \\ \delta q_1 \\ \delta q_2 \\ \delta q_3 \end{bmatrix}, \quad \delta\omega_b = \begin{bmatrix} \delta\omega_{bx} \\ \delta\omega_{by} \\ \delta\omega_{bz} \end{bmatrix}, \quad \delta\omega_n = \begin{bmatrix} \delta\omega_{nx} \\ \delta\omega_{ny} \\ \delta\omega_{nz} \end{bmatrix}$$
$$= \begin{bmatrix} \delta\Omega_x + \delta\rho_x \\ \delta\rho_y \\ \delta\Omega_z + \delta\rho_z \end{bmatrix}$$
$$= \begin{bmatrix} -|\omega_e|\sin\lambda^*\delta\lambda + \delta V_y/(R_E+h)^* \\ -V_y^*\delta h/[R_E+h)^*]^2 \\ -\delta V_x/(R_N+h)^* + V_x^*\delta h/[(R_N+h)^*]^2 \\ -(|\omega_e|\cos\lambda^* + \dot{\Lambda}^*\sec\lambda^*)\delta\lambda - \tan\lambda^*\delta V_y \\ \div (R_E+h)^* + \dot{\Lambda}^*\sin\lambda^*\delta h(R_E+h)^* \end{bmatrix} \tag{15}$$

Similarly, velocity errors satisfy the next differential equations:

$$\delta\dot{V}_N = [(\partial C_B^N/\partial q^*)\cdot\delta q]a_b^* + C_B^N(q^*)\delta a_b - [(2\delta\Omega+\delta\rho)_n]V_N^*$$
$$- [(2\Omega_e^*+\rho^*)_n]\delta V_N + \delta g$$
$$= 2C_B^N(q^*)A^{*T}\cdot O^{*T}\cdot\delta q + C_B^N(q^*)\delta a_b - [(2\delta\Omega+\delta\rho)_n]V_N^*$$
$$- [(2\Omega_e^*+\rho^*)_n]\delta V_N + \delta g$$
$$= C_B^N(q^*)(\delta a_b + 2\cdot A^{*T}\cdot O^{*T}\delta q) - [(2\delta\Omega+\delta\rho)_n]V_N^*$$
$$- [(2\Omega_e^*+\rho^*)_n]\delta V_N + \delta g \tag{16}$$

where

$$A^* = \begin{bmatrix} -a_{bx}^* & -a_{by}^* & -a_{bz}^* \\ 0 & -a_{bz}^* & -a_{by}^* \\ a_{bz}^* & 0 & -a_{bx}^* \\ -a_{by}^* & a_{bx}^* & 0 \end{bmatrix} = \begin{bmatrix} -a_b^{*T} \\ \hline [a_b^*]^T \end{bmatrix}$$

$$O^* = [-q^* \,|\, Q(q^*)] = \begin{bmatrix} -q_0^* & -q_1^* & -q_2^* & -q_3^* \\ -q_1^* & q_0^* & -q_3^* & q_2^* \\ -q_2^* & q_3^* & q_0^* & -q_1^* \\ -q_3^* & -q_2^* & q_1^* & q_0^* \end{bmatrix} \tag{17}$$

In Eq. (16), $(\delta a_b + 2\cdot A^{*T}\cdot O^{*T}\cdot\delta q)$ is interpreted as the total effective acceleration error vector in the body-fixed coordinates. The second term is the contribution to the total error induced by the attitude error. Upon use of Eq. (17), this term becomes

$$2\cdot A^{*T}\cdot O^{*T}\delta q = A^{*T}\begin{bmatrix} -2\cdot q^{*T}\cdot\delta q \\ \hline 2\cdot Q(q^*)^T\delta q \end{bmatrix} = c\cdot a_b^* + a_b^* x\phi$$

where

$$c = 2 \cdot q^{*T} \cdot \delta q = \delta(q^{*T} \cdot q^*)$$

$$\phi = -2 \cdot Q(q^*)^T \cdot \delta q \qquad (\because [a_b] = -(a_b x)) \tag{18}$$

Thus, the total acceleration error vector is

$$\delta\alpha = \delta a_b + a_b^* x\phi + c \cdot a_b^* \tag{19}$$

The terms δa_b and $a_b^* x\phi$ appear in the error analysis of the gimbaled platform system, where ϕ represents the vector of infinitesimal error angles (tilts) about the local vertical coordinates. Accordingly, the vector ϕ defined by Eq. (18) can be interpreted as the equivalent tilt in strapdown navigation systems. The third term, $c \cdot a_b^*$, in Eq. (19) represents a sale factor error, with scale factor c being the error in the square of the length of the vector q. This error is avoided by normalizing the quaternion vector to a unit vector.

$$q_c = q^* / \sqrt{q^{*T} \cdot q^*} \tag{20}$$

If, however, the tilt is produced by angular velocity error only, equivalent tilt ϕ satisfies the next differential equation.

$$\dot\phi = -2 \cdot Q(q^*)^T \delta\dot q - 2 \cdot \dot Q(q^*)^T \delta q$$

$$= -Q(q^*)^T \cdot ([\Omega_b^*]\delta q - [\Omega_n^*]\delta q + Q(q^*)\delta\omega_b - R(q^*)\delta\omega_n)$$

$$+ (Q(q^*)^T \cdot [\Omega^*]^T + 2(\Delta\omega^*) q^{*T})\delta q = -Q(q^*) \cdot [\Omega_b^*]\delta q$$

$$+ Q(q^*)^T \cdot [\Omega_n^*]\delta q - Q(q^*)^T \cdot Q(q^*)\delta\omega_b$$

$$+ Q(q^*)^T \cdot R(q^*)\delta\omega_n - Q(q^*)^T \cdot [\Omega^*]\delta q + 2(\Delta\omega^*) q^{*T}\delta q \tag{21}$$

where

$$\Delta\omega^* = \begin{bmatrix} \Delta\omega_x^* \\ \Delta\omega_y^* \\ \Delta\omega_z^* \end{bmatrix} = \begin{bmatrix} \omega_{bx}^* \\ \omega_{by}^* \\ \omega_{bz}^* \end{bmatrix} - C_N^B(q^*) \begin{bmatrix} \omega_{nx}^* \\ \omega_{ny}^* \\ \omega_{nz}^* \end{bmatrix} \tag{22}$$

$$[\Omega^*] = \left[\begin{array}{c|ccc} 0 & -\Delta\omega_x & -\Delta\omega_y & -\Delta\omega_z \\ \hline \Delta\omega_x & 0 & \Delta\omega_z & -\Delta\omega_y \\ \Delta\omega_y & -\Delta\omega_z & 0 & \Delta\omega_x \\ \Delta\omega_z & \Delta\omega_y & -\Delta\omega_x & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} 0 & -\Delta\omega^{*T} \\ \hline \Delta\omega^* & [\Delta\omega^*] \end{array}\right] \tag{23}$$

From Eqs. (7) and (18),

$$Q(q^*)\phi = -2 \cdot Q(q^*) \cdot Q(q^{*T}) \cdot \delta q = -2(I - q^* \cdot q^{*T}),$$

$$(\because O(q^*)O(q^*)^T = I)$$

Therefore,

$$\delta q = -Q(q^*) \cdot \phi/2 + q^* \cdot q^{*T} \cdot \delta q \tag{24}$$

Now, substitute into Eq. (21) this expression for δq to obtain

$$\dot\phi = -Q(q^*)^T [\Omega_b^*](-Q(q^*)\phi/2 + q^* \cdot q^{*T}\delta q)$$

$$+ Q(q^*)^T [\Omega_n^*](-Q(q^*)\phi/2 + q^* \cdot q^{*T}\delta q)$$

$$-\delta\omega_b + C_N^B(q^*)\delta\omega_n - Q(q^*)^T [\Omega^*]$$

$$\times (-Q(q^*)\phi/2 + q^* \cdot q^{*T}\delta q) + 2(\Delta\omega^*) q^{*T} \cdot \delta q$$

$$= ½ \cdot Q(q^*)^T [\Omega_b^*] Q(q^*)\phi - ½ \cdot Q(q^*)^T [\Omega_n^*] Q(q^*)\phi$$

$$+ ½ \cdot Q(q^*)^T [\Omega^*] Q(q^*)\phi - Q(q^*)^T [\Omega_b^*] q^* \cdot q^{*T} \cdot \delta q$$

$$+ Q(q^*)^T [\Omega_n^*] q^* \cdot q^{*T} \cdot \delta q - Q(q^*)^T [\Omega^*] q^* \cdot q^{*T} \cdot \delta q$$

$$+ 2 \cdot (\Delta\omega^*) q^{*T} \cdot \delta q - \delta\omega_b + C_N^B(q^*)\delta\omega_n \tag{25}$$

where

$$½ \cdot Q(q^*)^T \cdot ([\Omega_b^*] - [\Omega_n^*]) Q(q^*)\phi$$

$$= ½ \cdot Q(q^*)^T \cdot [\Omega^*] Q(q^*)\phi$$

$$Q(q^*)^T \cdot [\Omega^*] Q(q^*)\phi = [\Delta\omega^*]\phi$$

$$Q(q^*) \cdot ([\Omega_b^*] q^* - [\Omega_n^*] q^*) q^{*T}\delta q$$

$$= Q(q^*)^T \cdot (Q(q^*)\omega_b^* - R(q^*)\omega_n^*) q^{*T} \cdot \delta q$$

$$= (\omega_b^* - C_N^B(q^*)\omega_n^*) q^{*T}\delta q = \Delta\omega^* \cdot q^{*T} \cdot \delta q$$

$$Q(q^*)^T \cdot [\Omega^*] q^* \cdot q^{*T} \cdot \delta q = Q(q^*)^T \cdot (Q(q^*)\omega_b^*$$

$$- R(q^*)\omega_n^*) q^{*T} \cdot \delta q = \Delta\omega^* \cdot q^{*T}\delta q$$

Thus, finally, we obtain

$$\dot\phi = [\Delta\omega^*]\phi - \delta\omega_b + C_N^B(q^*)\delta\omega_n$$

$$= -\Delta\omega^* x\phi - (\delta\omega_b - C_N^B(q^*)\delta\omega_n) \tag{26}$$

This is the equation obtained in Ref. 3 and has the same form as the equation for error propagation in a gimbaled system. It is readily shown that the position errors satisfy the following differential equations:

$$\delta\dot\lambda = \delta Vx/(R_N + h)^* - V_x^* \delta h / [(R_N + h)^*]^2$$

$$\delta\dot\Lambda = \sec\lambda^* \delta V_y / (R_E + h)^* + \dot\Lambda^* \tan\lambda^* \delta\lambda - \dot\Lambda^* \delta h / (R_E + h)^*$$

$$\delta\dot h = (\dot h - \dot h^*) = -\delta V_z \tag{27}$$

IV. Summary and Conclusion

An application of the theory of strapdown inertial navigation introduced by Friedland[1] is presented in local vertical navigation error equations. The resulting equations can be easily changed into the results obtained in Ref. 1 by untorquing the navigation coordinate and modification of gravity term. These equations will contribute greatly to the construction of optimal filters for hybrid navigation systems.

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